ANSWERS

FOR THE
IB DIPLOMA
PROGRAMME

Mathematics

APPLICATIONS AND INTERPRETATION SL

EXAM PRACTICE WORKBOOK

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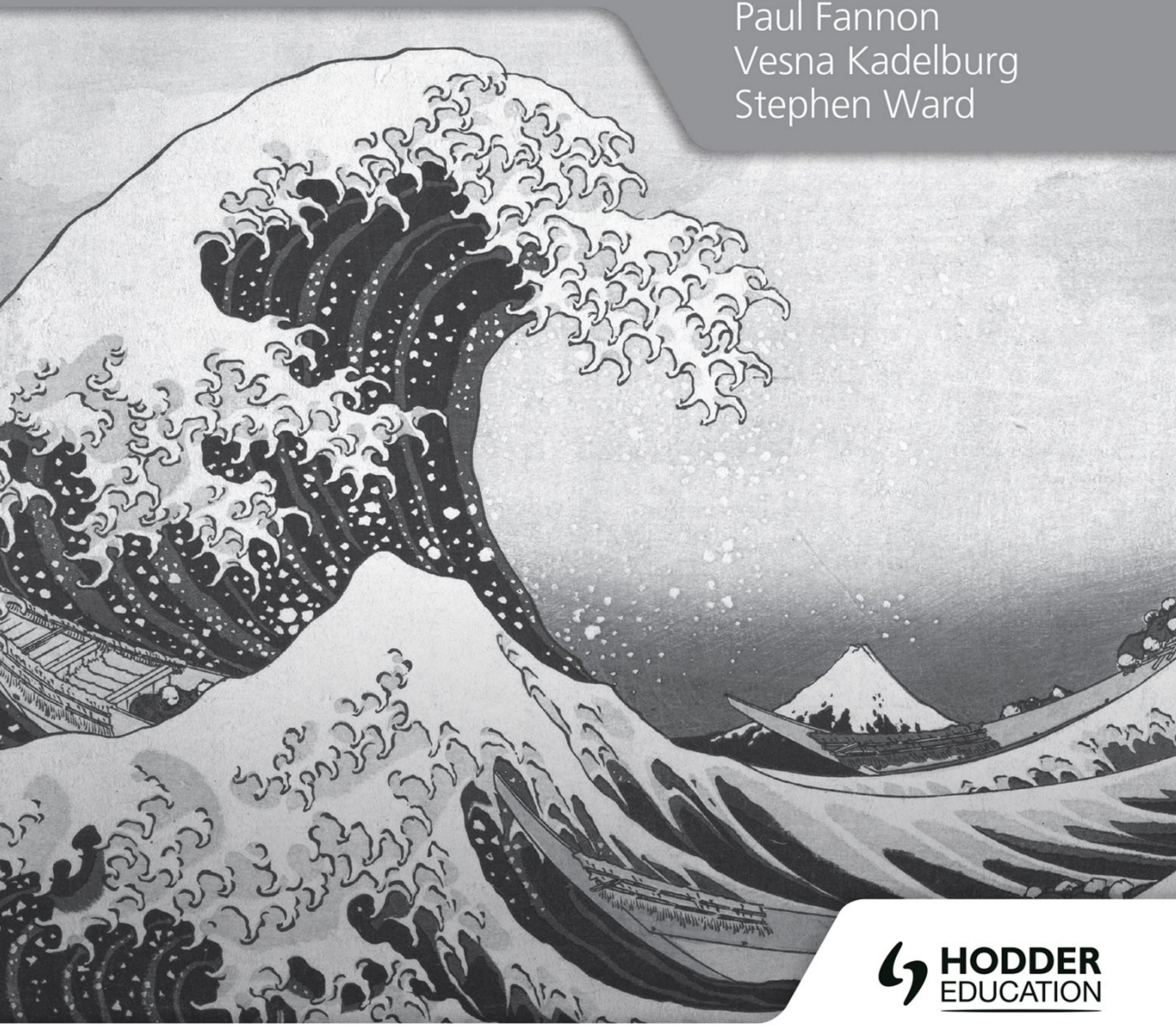


HODDER
EDUCATION

Answers to Practice Questions

1 Number and algebra

1 Just plug the numbers into your calculator. The answer is 4×10^{80}

2 $3 \times 10^{n+1} - 4 \times 10^n = 10^n(3 \times 10^1 - 4)$
$= 10^n(30 - 4)$
$= 26 \times 10^n$
$= 2.6 \times 10 \times 10^n$
$= 2.6 \times 10^{n+1}$

3 We can write this expression as

$$\left(\frac{6}{8}\right) \times \left(\frac{10^n}{10^{-n}}\right) = 0.75 \times 10^{n-(-n)}$$
$$= 7.5 \times 10^{-1} \times 10^{2n}$$
$$= 7.5 \times 10^{2n-1}$$

4 The first term is 20. The common difference is -3. Therefore

$$u_{25} = 20 + (-3) \times (25 - 1)$$
$$= 20 - 3 \times 24$$
$$= -52$$

Tip: You could have just used $u_1 = 20, u_{n+1} = u_n - 3$ in your calculator sequence function.

5 $u_n = 1602 = 21 + 17(n - 1)$

$1581 = 17(n - 1)$

$93 = n - 1$

$94 = n$

6 $u_4 = 10 = u_1 + 3d$

$u_{10} = 34 = u_1 + 9d$

Using the calculator to solve these equations simultaneous

$u_1 = -2$

$d = 4$

Therefore

$$u_{20} = -2 + 19 \times 4$$
$$= 74$$

7 $u_1 = 13, d = -3$ so

$$S_{30} = \frac{30}{2}\big(2 \times 13 - 3 \times (30 - 1)\big) = -915$$

8 $S_{20} = \frac{20}{2}(4 + 130) = 1340$

9 The easiest way to deal with sigma notation is to write out the first few terms by substituting in $r = 1, r = 2, r = 3$, etc.:

$S_n = 16 + 21 + 26 \ldots$

So the first term is 16 and the common difference is 5.

10 Your GDC should have a sum function, which can be used for this. The answer will be 26 350, but you should still write down the first term and common difference found in question 9 as part of your working.

11 This is an arithmetic sequence with first term 500 and common difference 100. The question is asking for S_{28}.

$$S_{28} = \frac{28}{2}(2 \times 500 + 100 \times 27) = 51\,800 \text{ m}$$

Tip: The hardest part of this question is realizing that it is looking for the sum of the sequence, rather than just how far Ahmed ran on the 28th day.

12 2.4% of \$300 is \$7.20. This is the common difference. After one year, there is \$307.20 in the account, so this is the 'first term'.

$u_{10} = 307.20 + 9 \times 7.20 = \$372$

Tip: The hardest part of this question is being careful with what 'after 10 years' means – it is very easy to be out by one year.

13 a The differences are 1.1, 0.8 and 0.8. Their average is 0.9. When $t = 0.5$ we are looking for the sixth term of the sequence, which would be

$u_6 = 0 + 0.9 \times 5 = 4.5 \text{ m s}^{-1}$

b There are many criticisms which could be made about this model – for example:

- There is too little data for it to be reliable.
- There is no theoretical reason given for it being an arithmetic sequence.
- The ball will eventually hit the ground.
- The model predicts that the ball's velocity grows without limit.
- There may be a pattern with smaller differences later on.

14 The first term is 32 and the common ratio is $-\frac{1}{2}$.

$$u_{10} = 32 \times \left(-\frac{1}{2}\right)^9 = -\frac{1}{16}$$

15 The first term is 1 and the common ratio is 2 so

$u_n = 1 \times 2^{n-1}$

If $u_n = 4096$ then $4096 = 2^{n-1}$

There are four ways you should be able to solve this:

- On a non-calculator paper you might be expected to figure out that $4096 = 2^{12}$.
- You can take logs of both side to get $\ln 4096 = (n - 1) \ln 2$ and solve for n.
- You can graph $y = 2^{x-1}$ and intersect it with $y = 4096$.
- You can create a table showing the sequence and determine which row 4096 is in.

Whichever way, the answer is 13.

16 $u_3 = u_1 r^2 = 16$

$u_7 = u_1 r^6 = 256$

Dividing the two equations:

$$\frac{u_1 r^6}{u_1 r^2} = \frac{256}{16}$$

$r^4 = 16$

$r = \pm 2$

$u_1 = \frac{16}{r^2} = 4$ for both possible values of r.

17 The first term is 162, the common ratio is $\frac{1}{3}$

$$S_8 = \frac{162\left(1 - \frac{1}{3}^8\right)}{1 - \frac{1}{3}} = \frac{6560}{27} \approx 243$$

Tip: You can always use either sum formula, but generally if r is between 0 and 1 the second formula avoids negative numbers.

18 The easiest way to deal with sigma notation is to write out the first few terms by substituting in $r = 1, r = 2, r = 3$, etc.:

$S_n = 10 + 50 + 250 \ldots$

So the first term is 10 and the common ratio is 5.

19 Your GDC should have a sum function, which can be used for this. The answer will be 24 414 060, but you should still write down the first term and common difference found in question 9 as part of your working.

20 a This is a geometric sequence with first term 50 000 and common ratio 1.2. ‘After 12 days’ corresponds to the 13th term of the sequence so:

$u_{13} = 50\,000 \times 1.2^{12} = 445\,805$

b The model suggests that the number of bacteria can grow without limit.

21 You could use the formula:

$$FV = 2000 \times \left(1 + \frac{4}{100 \times 12}\right)^{12\times 10} = 2981.67$$

However, the general expectation is that you would use the TVM package on your calculator for this type of question. Make sure you can get the same answer using your package as different calculators have slightly different syntaxes.

22 Using the TVM package, $i = 5.10\%$

23 Unless stated otherwise, you should assume that the compound interest is paid annually. Using $FV = 200$ and $PV = -100$, the TVM package suggests that 33.35 years are required. Therefore 33 years would be insufficient, so 34 complete years are required.

24 12% annual depreciation is modelled as compound interest with an interest rate of 12%. Using the TVM package the value is $24 000 to three significant figures.

25 When adjusting for inflation, the 'real' interest rate is $3.2 - 2.4 = 0.8\%$. Using this value in the TVM package gives a final value of \$2081.

26 This expression is $2^{-2\times-2} = 2^4 = 16$

27 $(2x)^3 = 2^3 \times x^3 = 8x^3$

28 This is equivalent to $x = \log_{10}(k)$

29 The given statement is equivalent to

$2x - 6 = \ln 5$

So

$2x = \ln 5 + 6$

$x = \frac{1}{2}\ln 5 + 3$

Tip: The answer could be written in several different ways – for example, $\ln(\sqrt{5}\, e^3)$. Generally, any correct and reasonably simplified answer would be acceptable.

30 Using appropriate calculator functions:

$\ln 10 + \log_{10} e \approx 2.30 + 0.434 \approx 2.74$

31 Look at the fourth decimal place. It is six, which is five or more, round up, so the answer is 0.011.

32 The fourth significant figure is zero, so round down to 105 000.

33 $500\,000 \times 0.1235 = 61\,750$; however, it is not appropriate to quote an answer to more than the least accurate original figure, so it should be reported as 60 000.

34 $12.445 \leq x < 12.455$

35 $\left|\frac{45-38}{38}\right| \times 100\% = 18.4\%$

36 The side of the square is between 6.5 and 7.5 cm. Therefore, the area is between 42.25 and 56.25 cm^2. The percentage errors for each of these are:

$\left|\frac{56.25-49}{56.25}\right| \times 100\% = 12.9\%$

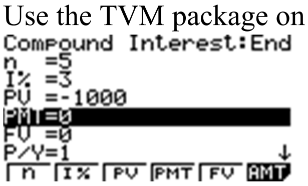
or

$\left|\frac{42.25-49}{42.25}\right| \times 100\% = 16.0\%$

so the largest possible error is 16.0%

37 Probabilities cannot be greater than 1.

38 Use the TVM package on your GDC:



Solve for the payment (PMT) variable, getting \$218.35

39 Using TVM with $PV = 100$, PMT = 100, the final value is \$1235.64

Tip: The calculator will give a negative answer – this is because it is treating it as money the bank owes you. You need to be able to interpret the outputs from the TVM package appropriately.

40 Setting $PV = \$50\,000$, $FV = 0$ and $n = 30$, solving for PMT gives a payment of \$2892.

41 Using the simultaneous equation solve on the calculator,

$x = 1, y = 0, z = -1$

42 Using the polynomial equation solve on your GDC:

$x = -0.303, 1$ or 3.30

2 Functions

1 Rearrange into the form $y = mx + c$:

$$3x - 4y - 5 = 0$$
$$4y = 3x - 5$$
$$y = \frac{3}{4}x - \frac{5}{4}$$

So, $m = \frac{3}{4}, c = -\frac{5}{4}$

2 Use $y - y_1 = m(x - x_1)$:

$$y + 4 = -3(x - 2)$$
$$y + 4 = -3x + 6$$
$$y = -3x + 2$$

3 Find the gradient using $m = \frac{y_2 - y_1}{x_2 - x_1}$:

$$m = \frac{1 + 5}{9 + 3} = \frac{1}{2}$$

Use $y - y_1 = m(x - x_1)$:

$$y - 1 = \frac{1}{2}(x - 9)$$
$$2y - 2 = x - 9$$
$$x - 2y - 7 = 0$$

4 Gradient of parallel line is $m = 2$

$$y - 4 = 2(x - 1)$$

$$y = 2x + 2$$

5 Gradient of perpendicular line is $m = -\frac{1}{-\frac{1}{4}} = 4$

$$y - 3 = 4(x + 2)$$

$$y = 4x + 11$$

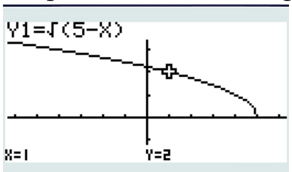
6 Substitute $x = -2$ into the function:

$$f(-2) = 3(-2)^2 - 4$$
$$= 8$$

7 $2x - 1 > 0$

$$x > \frac{1}{2}$$

8 Graph the function using the GDC:

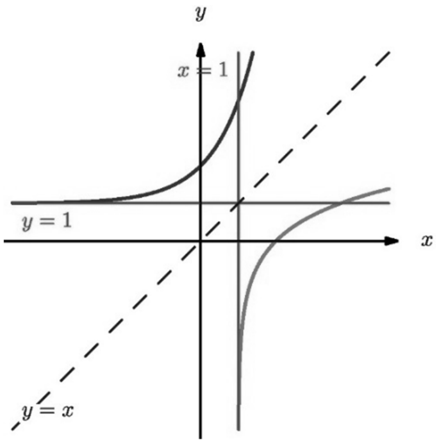


$f(1) = 2$, so range is $f(x) \geq 2$

9 To find $f^{-1}(-8)$, solve $f(x) = -8$:

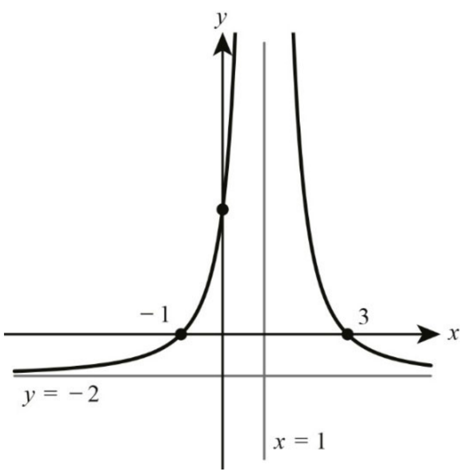
$$4 - 3x = -8$$
$$-3x = -12$$
$$x = 4$$

10 Reflect the graph in the line $y = x$.

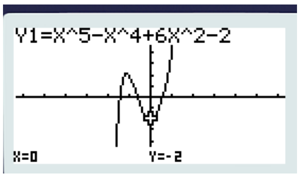


11 Put in the vertical and horizontal asymptote and the x-intercepts (zeros of the function).

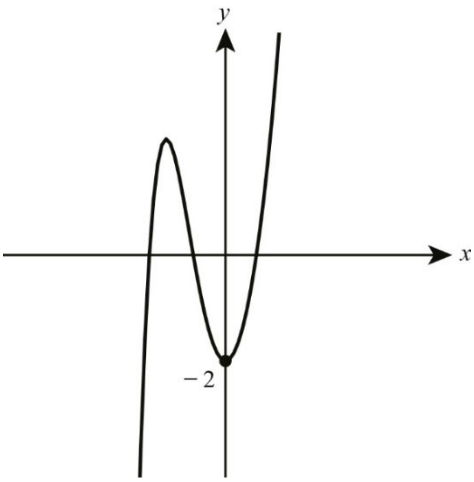
Since $f(x) > -2$, it must tend to ∞ as it approaches the vertical asymptote from either side.



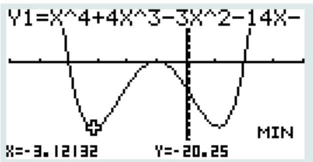
12 Graph the function using the GDC:



The y-intercept is $(0, -2)$. Now sketch the graph from the plot on the GDC:



13 a Graph the function and use 'min' and 'max' to find the coordinates of the vertices, moving the cursor as necessary:

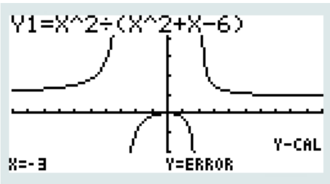


Coordinates of vertices: $(-3.12, -20.3), (-1, 0), (1.12, -20.3)$

b From the graph you can see there is a line of symmetry through the maximum point:

Line of symmetry: $x = -1$

14 Graph the function and look for values where there appear to be asymptotes:



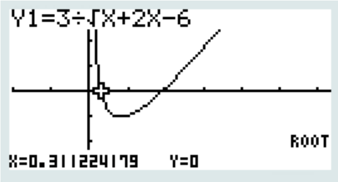
Vertical asymptotes occur at values of x where the y values appears as 'error':

Vertical asymptotes: $x = -3$ and $x = 2$

The y-value approaches 1 as the x-value gets big and positive or big and negative:

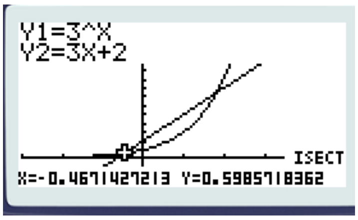
Horizontal asymptote: $y = 1$

15 Graph the function and use ‘root’, moving the cursor from one to the other:



Zeros: $x = 0.311, 1.92$

16 Graph the function and use ‘isct’, moving the cursor from one intersection point to the other:



Points of intersection: $(-0.467, 0.599)$ and $(1.83, 7.50)$

17 Substitute each pair of conditions into the model:

$$\begin{cases} m(-5) + c = 10 \\ m(1) + c = -8 \end{cases}$$

$$\begin{cases} -5m + c = 10 \\ m + c = -8 \end{cases}$$

Solve simultaneously using the GDC: $m = -3, c = -5$

So, $\mathrm{f}(x) = -3m - 5$

18 a Substitute $x = 3$ into the function as defined on the domain $0 \leq x \leq 10$:

$$\begin{aligned} \mathrm{f}(3) &= 0.5(3) + 11 \\ &= 12.5 \end{aligned}$$

b Substitute $x = 11$ into the function as defined on the domain $x > 10$:

$$\begin{aligned} \mathrm{f}(11) &= 2(11) - 4 \\ &= 18 \end{aligned}$$

19 Substitute the three sets of conditions into the model:

$$\begin{cases} a(1)^2 + b(1) + c = -3 \\ a(2)^2 + b(2) + c = 4 \\ a(3)^2 + b(3) + c = 17 \end{cases}$$

$$\begin{cases} a + b + c = -3 \\ 4a + 2b + c = 4 \\ 9a + 3b + c = 17 \end{cases}$$

Solve simultaneously using the GDC: $a = 3, b = -2, c = -4$

So, $\mathrm{f}(x) = 3x^2 - 2x - 4$

20 The x-coordinate of the vertex is $x = -\frac{b}{2a}$:

$$-2 = -\frac{b}{2a} \Rightarrow 4a = b$$

The y-intercept is $x = 0, y = 9$:

$c = 9$

Substitute $x = 1, y = 19$ into $y = ax^2 + bx + 9$:

$$19 = a + b + 9$$
$$a + b = 10$$

Since $b = 4a$,

$$a + 4a = 10$$
$$a = 2$$

So, $y = 2x^2 + 8x + 9$

21 Substitute each pair of conditions into the model:

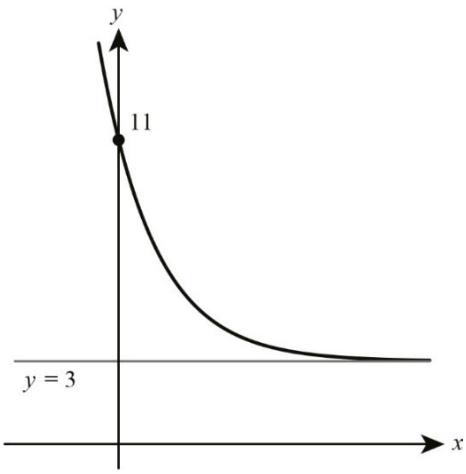
$$\begin{cases} k \times 3^{-(-2)} + c = 38 \\ k \times 3^{-(-1)} + c = 14 \end{cases}$$

$$\begin{cases} 9k + c = 38 \\ 3k + c = 14 \end{cases}$$

Solve simultaneously using the GDC: $k = 4, c = 2$

So, $\mathrm{f}(x) = 4 \times 3^{-x} + 2$

22 Plot on the GDC and identify the horizontal asymptote at $y = 3$:

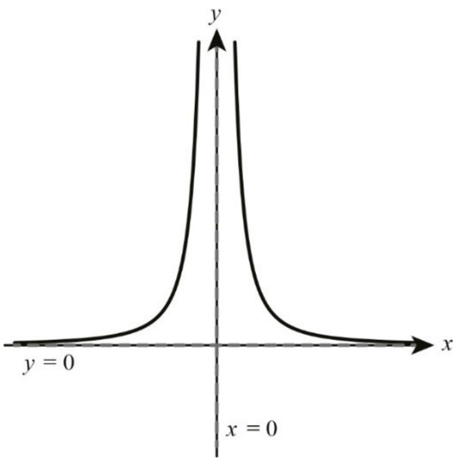


23 Substitute the given condition into $y = \frac{k}{x^2}$:

$$3 = \frac{k}{2^2}$$
$$k = 12$$

So, $y = \frac{12}{x^2}$

24 Plot on the GDC and identify the horizontal and vertical asymptotes:



Horizontal asymptote: $y = 0$

Vertical asymptote: $x = 0$

25 Substitute the four sets of conditions into the model:

$$\begin{cases} a(-2)^3 + b(-2)^2 + c(-2) + d = 1 \\ a(-1)^3 + b(-1)^2 + c(-1) + d = 7 \\ a(1)^3 + b(1)^2 + c(1) + d = 1 \\ a(2)^3 + b(2)^2 + c(2) + d = 13 \end{cases}$$

$$\begin{cases} -8a + 4b - 2c + d = 1 \\ -a + b - c + d = 7 \\ a + b + c + d = 1 \\ 8a + 4b + 2c + d = 13 \end{cases}$$

Solve simultaneously using the GDC: $a = 2, b = 1, c = -5, d = 3$

So, $\mathrm{f}(x) = 2x^3 + x^2 - 5x + 3$

26 The amplitude is a:

$a = 4$

The period is $\frac{360}{b}$:

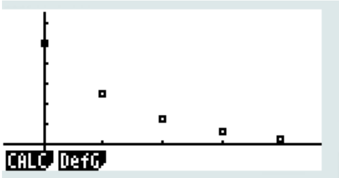
$$\frac{360}{b} = 240$$
$$b = \frac{3}{2}$$

The principal axis is $y = d$:

$d = -1$

So, $\mathrm{f}(x) = 4\sin\left(\frac{3}{2}x\right) - 1$

27 Plot the data on the GDC to see which model the graph most closely resembles:



The graph seems to have a horizontal asymptote at $y = 0$ but not a vertical asymptote at $x = 0$, so $y = k \times 2^{rx}$ is the most appropriate model.

28 There are two parameters to find (k and r) so form two equations using the first two points:

$$\begin{cases} k \times 2^{r\times 0} = 5 \\ k \times 2^{r\times 1} = 2.5 \end{cases}$$

$$\begin{cases} k = 5 \\ k \times 2^{r} = 2.5 \end{cases}$$

$$\begin{aligned} 5 \times 2^{r} &= 2.5 \\ 2^{r} &= \frac{1}{2} \\ r &= -1 \end{aligned}$$

So, $y = 5 \times 2^{-x}$

29 The market share cannot exceed 100%:

$$\begin{aligned} &3x + 4 \leq 100 \\ &x \leq 32 \end{aligned}$$

Also, the amount spent cannot be negative, so a suitable domain is $0 \leq x \leq 32$.

30 The units of x are thousands of \$, so substitute $x = 18$ into the function:

$$\begin{aligned} s(18) &= 3 \times 18 + 4 \\ &= 58\% \end{aligned}$$

31 A 100% market share (or anything close) is highly unlikely to be achievable.

The increase in market share is highly unlikely to increase linearly with extra spending on marketing – it is much more likely there will be an ever slower rate of increase.

32 An exponential model tending to an asymptote below 100%, e.g. of the form $y = c - a^{-x}$, where $c < 100$.

3 Geometry and trigonometry

1 $$\begin{aligned} d &= \sqrt{(7-2)^2 + (3-(-4))^2 + (-1-5)^2} \\ &= \sqrt{25 + 49 + 36} \\ &= 10.5 \end{aligned}$$

2 $$\begin{aligned} M &= \left(\frac{1+(-5)}{2}, \frac{8+2}{2}, \frac{-3+4}{2}\right) \\ &= (-2, 5, 0.5) \end{aligned}$$

3 Radius = 8 cm

Volume $= \frac{4}{3}\pi r^3 \quad = \frac{4}{3}\pi \times 8^3$

$= 2140\ \text{cm}^3$

Surface area $= 4\pi r^2 \ = 4\pi \times 8^2$

$= 804\ \text{cm}^2$

4 Volume $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times 6^2 \times 15$

$= 565\ \text{cm}^3$

Slope length, l, is given by:

$$\begin{aligned} l &= \sqrt{r^2 + h^2} \\ &= \sqrt{6^2 + 15^2} \\ &= 3\sqrt{29} \end{aligned}$$

Surface area $= \pi r l + \pi r^2$

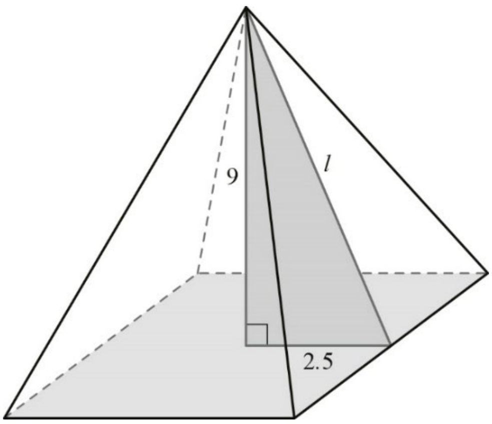
$= \pi \times 6 \times 3\sqrt{29} + \pi \times 6^2$

$= 418\ \text{cm}^2$

5 Volume $= \frac{1}{3}x^2 h$

$= \frac{1}{3} \times 5^2 \times 9$

$= 75.0\ \text{cm}^3$



$$\begin{aligned} l &= \sqrt{2.5^2 + 9^2} \\ &= \frac{\sqrt{349}}{2} \end{aligned}$$

Surface area $= 5^2 + 4\left(\frac{1}{2} \times 5 \times \frac{\sqrt{349}}{2}\right)$

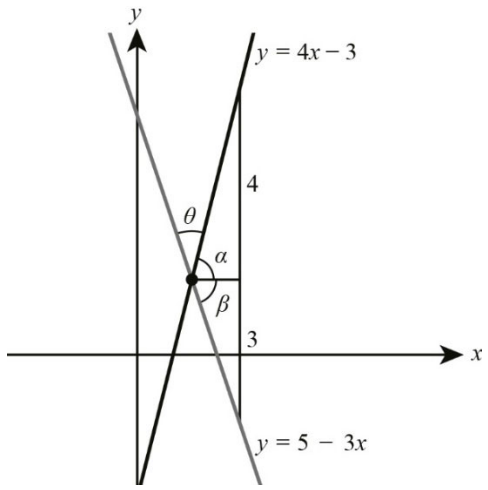
$= 118\ \text{cm}^2$

6 Volume $= \pi r^2 h + \frac{2}{3}\pi r^3$

$= \pi \times 5^2 \times 30 + \frac{2}{3}\pi \times 5^3$

$= 2620\ \text{m}^3$

7 Draw the lines and label the angle required θ:



Since the gradient of $y = 4x - 3$ is 4,

$\tan\alpha = \frac{4}{1}$

$\alpha = \tan^{-1} 4 = 76.0°$

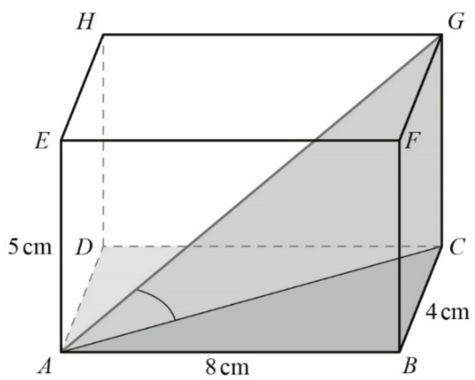
Since the gradient of $y = 5 - 3x$ is -3,

$\tan\beta = \frac{3}{1}$

$\alpha = \tan^{-1} 3 = 71.6°$

So, $\theta = 180 - 76.0 - 71.6 = 32.4°$

8 Draw in the diagonal AG. Angle needed is $G\hat{A}C = \theta$



First work in triangle ABC:

$AC = \sqrt{8^2 + 4^2}$

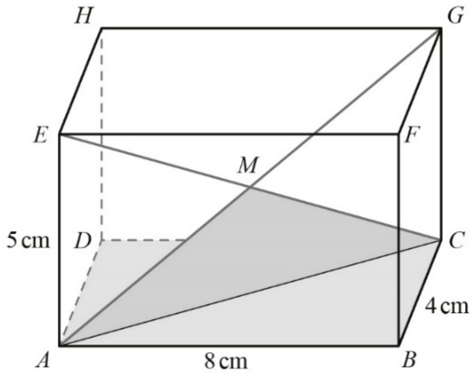
$= \sqrt{80}$

Then in triangle ACG:

$\tan\theta = \frac{5}{\sqrt{80}}$

$\theta = \tan^{-1}\frac{5}{\sqrt{80}} = 29.2°$

9 Draw in the two diagonals – they will intersect at the midpoint of each, M.



From triangle AGC:

$$AG = \sqrt{\left(\sqrt{80}\right)^2 + 5^2}$$
$$= \sqrt{105}$$

By symmetry, $EC = \sqrt{105}$

And $AM = CM = \frac{\sqrt{105}}{2}$

Using the cosine rule in triangle AMC:

$$\cos M = \frac{AM^2 + CM^2 - AC^2}{2(AM)(CM)}$$
$$= \frac{26.25 + 26.25 - 80}{52.5}$$
$$M = 121.6°$$

So, acute angle between AG and EC is $180 - 121.6 = 58.4°$

10 $\sin\theta = \frac{1.8}{4.9}$

$$\theta = \sin^{-1}\frac{1.8}{4.9}$$
$$= 21.6°$$

11 By the sine rule,

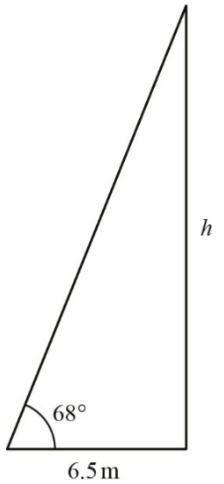
$$\frac{3.8}{\sin 80} = \frac{AC}{\sin 55}$$
$$AC = \frac{3.8}{\sin 80} \times \sin 55$$
$$= 3.16 \text{ cm}$$

12 By the cosine rule,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$= \frac{12^2 + 10^2 - 9^2}{2 \times 12 \times 10}$$
$$C = \cos^{-1}\left(\frac{12^2 + 10^2 - 9^2}{2 \times 12 \times 10}\right) = 47.2°$$

13 $A = \frac{1}{2} \times 6 \times 15 \times \sin 42$
$= 30.1$

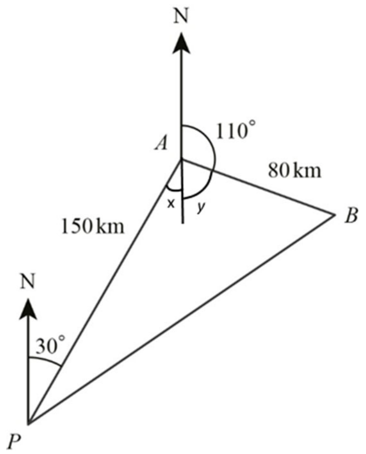
14 Draw a diagram:



$$\tan 68 = \frac{h}{6.5}$$

$$h = 6.5 \tan 68$$
$$= 16.1 \text{ m}$$

15 Start by drawing the situation described:



$x = 30°$ by alternate angles

$y = 180 - 110 = 70°$

So, $P\hat{A}B = 30 + 70 = 100°$

By the cosine rule,

$$d^2 = 150^2 + 80^2 - 2 \times 150 \times 80 \cos 100$$
$$d = \sqrt{150^2 + 80^2 - 2 \times 150 \times 80 \cos 100}$$
$$= 182 \text{ km}$$

16 $s = \frac{\theta}{360} \times 2\pi r$

$$= \frac{55}{360} \times 2\pi \times 6$$
$$= 5.76 \text{ cm}$$

17 $A = \frac{\theta}{360} \times \pi r^2$

$= \frac{110}{360} \times \pi \times 10^2$

$= 96.0 \text{ cm}^2$

18 Rearrange the equation to find the gradient of the line segment AB:

$$2x + 3y = 5$$
$$3y = -2x + 5$$
$$y = -\frac{2}{3}x + \frac{5}{3}$$

The gradient of AB is $-\frac{2}{3}$

So the gradient of the perpendicular to AB is $-\frac{1}{\left(-\frac{2}{3}\right)} = \frac{3}{2}$

The perpendicular bisector has gradient $\frac{3}{2}$ and passes through the point $(4, 7)$:

$$y - 7 = \frac{3}{2}(x - 4)$$

$$y = \frac{3}{2}x + 1$$

19 The midpoint of $(-3, -2)$ and $(1, 8)$ is $\left(\frac{-3+1}{2}, \frac{-2+8}{2}\right) = (-1, 3)$

The gradient of the line segment between $(-3, -2)$ and $(1, 8)$ is $\frac{8-(-2)}{1-(-3)} = \frac{5}{2}$

So the gradient of the perpendicular bisector is $-\frac{1}{\left(\frac{5}{2}\right)} = -\frac{2}{5}$

The perpendicular bisector has gradient $-\frac{2}{5}$ and passes through the point $(-1, 3)$:

$$y - 3 = -\frac{2}{5}(x + 1)$$

$$y = -\frac{2}{5}x + \frac{13}{5}$$

20 a The sites are the labelled points:

A, B, C, D, E

b The vertices are the points where the edges meet:

P, Q, R, S, T

c The finite edges are the lines that have vertices at both ends:

PQ, QR, RS, ST, PT, QT

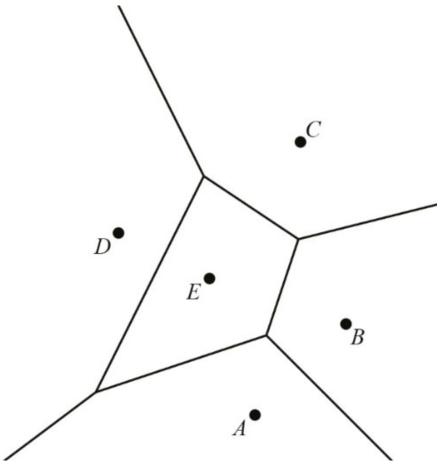
d The finite cells are the fully enclosed areas:

$PTQ, TQRS$

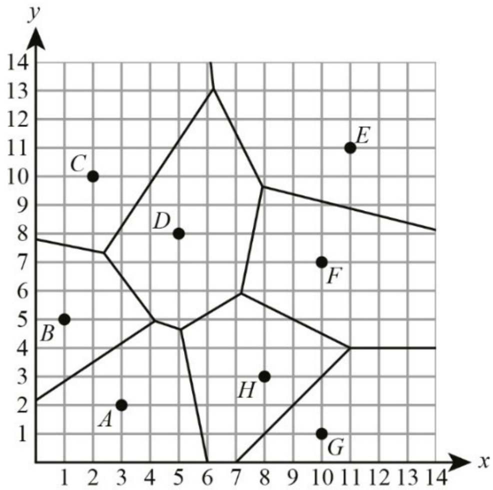
21 Apply the incremental algorithm starting with the perpendicular bisector of DE, since E is in cell D.

Then travel along the perpendicular bisector of CE, them BE, then AE.

That completes the cell E so delete all unused perpendicular bisectors:



22



$(5, 4)$ lies in cell A, so by nearest neighbour interpolation, estimate the value of the function to be 12.

23 The possible locations are the four vertices. Of these, the one furthest from any town is clearly CDE.

Find the coordinates of vertex CDE as the intersection of the perpendicular bisectors of CD and DE.

Midpoint of C and D is $(6, 7)$ and the perpendicular between them has gradient -0.5.

Bisector CD has equation:

$y - 7 = -0.5(x - 6)$
$2y + x = 20$ **(1)**

Midpoint of D and E is $(6.5, 3)$ and the perpendicular between them has gradient 0.75.

Bisector of DE has equation

$y - 3 = 0.75(x - 6.5)$
$6x - 8y = 15$ **(2)**

The intersection of these lines is vertex CDE:

$(2) + 4(1): 10x = 95$

CDE has coordinates $(9.5, 5.25)$

4 Statistics and probability

1 a Discrete – it can only take certain values.

b Continuous (although its measurement might be discrete because it will be to a particular accuracy).

c Discrete.

2 a There are many possibilities. It could be all his patients, all his ill patients, all people in his area, or all people in the world.

b This is not a random sample of the population because there are some people (those who do not attend the clinic) who cannot possibly be included.

3 Yes – people who do less exercise might be less likely to choose to participate.

4 Yes – there seems to be consistency when the observation is repeated (within a reasonable statistical noise).

5 Item D is not a possible human height. It might have been a participant not taking the test seriously or it might have been a misread (e.g. giving height in feet and inches). You could either return to participant D and ask them to check their response, or if this were not possible you would discard the data item.

6 The IQR is 4. Anything above $11 + 1.5 \times 4 = 17$ is an outlier. Just because a data item is an outlier does not mean that it should be excluded. It should be investigated carefully to ensure that it is still a valid member of the population of interest.

7 Convenience sampling.

8 The proportion from Italy is $\frac{60}{150} = \frac{2}{5}$. The stratified sample must be in the same proportion, so it should contain $\frac{2}{5} \times 20 = 8$ students from Italy.

9 The proportion is $\frac{18+12}{15+18+12} = \frac{30}{45} = \frac{2}{3}$

10 This is all the 30–40 group and half of the 20–30 group, which is $7+\frac{4}{2} = 9$

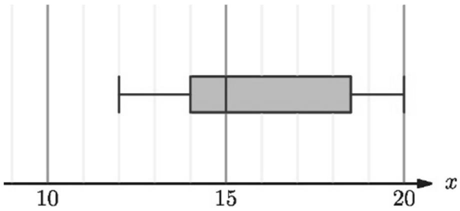
11 a The total frequency is 160. Half of this frequency (80) on the frequency axis reads as about 42 on the x-axis, which is the median.

b The lower quartile corresponds to a frequency of 40, which is an x-value of approximately 30.

The upper quartile corresponds to a frequency of 120 which is an x-value of approximately 60. Therefore, the interquartile range is $60 - 30 = 30$.

c The 90th percentile corresponds to a frequency of $0.9 \times 160 = 144$. This has an x-value of about 72, which is the 90th percentile.

12 Putting the data into the GDC, the following summary statistics can be found:



Min 12; lower quartile 14; median 16; upper quartile 18.5; max 20

13 a Both have a similar spread (same IQR (4) and range excluding outliers (10)) but population A is higher on average (median of 16 versus 10).

b B is more likely to be normally distributed as it has a symmetric distribution.

14 The mode is 14 as it is the only one which occurs twice. From the GD, the median is 18 and the mean is 18.5

Tip: You should also be able calculate the median without a calculator.

15 The mean is $\frac{23+x}{5} = 7$

So $23 + x = 35$, therefore $x = 12$

16 a The midpoints are 15, 25, 40, 55.

$n = 10 + 12 + 15 + 13 = 50$

$$\bar{x} = \frac{10 \times 15 + 12 \times 25 + 15 \times 40 + 13 \times 55}{50} = \frac{1765}{50} = 35.3$$

b We are not using the original data.

17 The modal class is $15 < x \leq 20$

18 a From the GDC, $Q_3 = 18, Q_1 = 7$ so IQR $= 11$

b From standard deviation is 5.45

c The variance is $5.45^2 = 29.7$

19 The new mean is $12 \times 2 + 4 = 28$

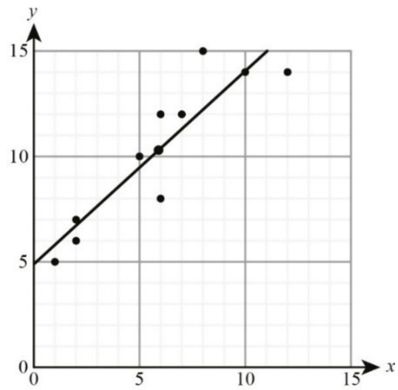
The new standard deviation is $10 \times 2 = 20$

20 Using the GDC, the lower quartile is 16 and the upper quartile is 28.5

21 a From the GDC: $r = 0.910$

b There is strong positive correlation between x and y.

22 a Something like:



b Approximately 7.5.

23 From GDC, $y = 0.916x + 4.89$

24 a i 13.1

ii 23.2

iii 5.58

b Only part **i**. Part **ii** is extrapolation and part **iii** is using a y-on-x line inappropriately.

Tip: A y-on-x regression line can only be used to predict values of y for given values of x; not the other way around.

25 a This is the expected number of text messages sent by a pupil who does not spend any time on social media in a day.

b For every additional hour spent on social media, the model predicts that the pupil will send 1.4 additional texts.

26 a Split the data into the first four points and the next five points and do a regression for each part separately.

$$L = \begin{cases} 4.19A - 0.259 & A < 6 \\ 0.830A + 25.4 & A > 6 \end{cases}$$

b Using the first part of the piecewise graph:

$L = 4.19 \times 3 - 0.259 = 12.3$

So expect a length of 12.3 cm.

27 $\frac{134}{200} = 0.67$

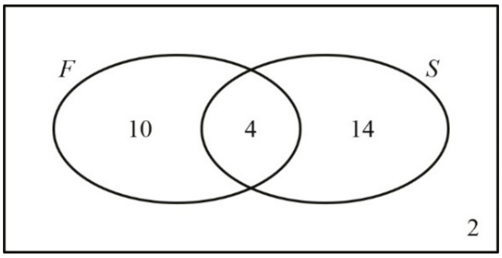
28 There are six possible outcomes of which three (2, 3 and 5) are prime, so the probability is $\frac{3}{6} = 0.5$

29 $\mathrm{P}(A') = 1 - \mathrm{P}(A) = 0.4$

30 $30 \times 0.05 = 1.5$

Tip: Remember that expected values should not be rounded to make them achievable.

31 We can illustrate this in a Venn diagram:



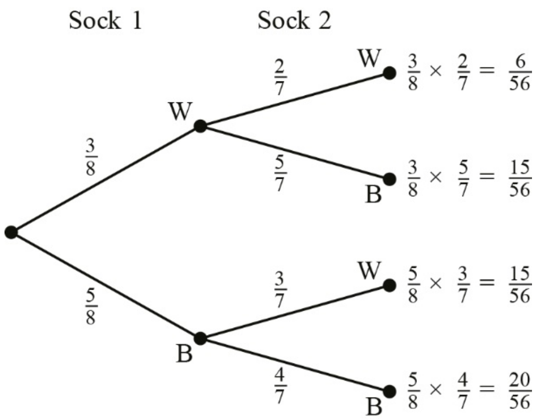
There are $14 - 4 = 10$ students who study only French.

There are $18 - 4 = 14$ students who study only Spanish.

Therefore, there are $10 + 14 + 4 = 28$ students who study either French or Spanish. This leaves 2 students who do not study either, so the probability is $\frac{2}{30} = \frac{1}{15}$

32 a There will be 7 socks left, of which 5 are black so it is $\frac{5}{7}$

b This is best illustrated using a tree diagram:



There are two branches relevant to the question.

White then black:

$$\frac{3}{8} \times \frac{5}{7} = \frac{15}{56}$$

Black then white:

$$\frac{5}{8} \times \frac{3}{7} = \frac{15}{56}$$

The total probability is $\frac{30}{56} = \frac{15}{28}$

33 a This can be best illustrated using a sample space diagram:

		1st roll			
		1	**2**	**3**	**4**
2nd roll	**1**	2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

There are 16 places in the sample space diagram. 6 of them have a score above 5 (shaded blue in the diagram) therefore the probability is $\frac{6}{16}=\frac{3}{8}$

b In the sample space diagram there are 6 scores above 5. Two of them are 7 so $\mathrm{P}(\text{score}=7|\text{score}>5)=\frac{2}{6}=\frac{1}{3}$

34 a $x=100-40-30-20=10$

b There are 60 out of 100 students who prefer maths, so $\frac{60}{100}=\frac{3}{5}$

35 $\mathrm{P}(A\cup B)=0.5+0.7-0.3=0.9$

36 $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-0=0.6$

37 There are 70 people who prefer soccer. Out of these 40 prefer maths. So $P(\text{maths}|\text{soccer})=\frac{40}{70}=\frac{4}{7}$

38 $\mathrm{P}(A\cap B)=\mathrm{P}(A)\mathrm{P}(B)=0.24$

39 W can take three possible values: 0, 1 or 2.

$$\mathrm{P}(W=0)=\mathrm{P}(BB)=\frac{4}{7}\times\frac{3}{6}=\frac{2}{7}$$

$$\mathrm{P}(W=1)=\mathrm{P}(BW)+\mathrm{P}(WB)=\frac{4}{7}\times\frac{3}{6}+\frac{3}{7}\times\frac{4}{6}=\frac{4}{7}$$

$$\mathrm{P}(W=2)=\mathrm{P}(WW)=\frac{3}{7}\times\frac{2}{6}=\frac{1}{7}$$

So

w	0	1	2
$\mathrm{P}(W=w)$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

40 We can create a table:

x	0	1	2
$\mathbf{P}(X = x)$	k	$2k$	$3k$

The total probability is $k + 2k + 3k = 6k$, which must equal 1 so $k = \frac{1}{6}$

41 $\mathrm{E}(X) = 0.5 \times 0.5 + 1 \times 0.4 + 2.5 \times 0.1 = 0.9$

42 a The probability of winning any prize is $0.095 + 0.005 = 0.1$

The probability of winning \$2000 is 0.005, so the conditional probability is $\frac{0.005}{0.1} = 0.05$

b $\mathrm{E}(X) = 0 \times 0.9 + 10 \times 0.095 + 2000 \times 0.005 = 10.95$

$\mathrm{P}(X > 10.95) = \mathrm{P}(X = 2000) = 0.005$

43 $\mathrm{E}(X) = -1 \times 0.6 + 0 \times 0.3 + 0.1k = 0.1k - 0.6$

If the game is fair then $\mathrm{E}(X) = 0$ so:

$0.1k - 0.6 = 0$

$0.1k = 0.6$

$k = 6$

44 The outcome of each trial is not independent of the previous trial.

45 Your calculator should have two functions – one which finds $\mathrm{P}(X = x)$, which we will use in part **a** and one which finds $\mathrm{P}(X \leq x)$ which we will use in part **b**.

a $\mathrm{P}(X = 2) = 0.3456$

b To use the calculator, we need to write the given question into a cumulative probability:

$\mathrm{P}(X \geq 3) = 1 - \mathrm{P}(X \leq 2) = 1 - 0.683 = 0.317$

46 a If X is the number of heads then $\mathrm{E}(X) = np = 10 \times 0.6 = 6$

b $\mathrm{Var}(X) = 10 \times 0.6 \times 0.4 = 2.4$ so the standard deviation is $\sqrt{2.4} \approx 1.55$

47 We know that about 68% of the data occurs within one standard deviation of the mean, but this gives negative values of time which are not possible.

48 a It is a symmetric, bell-shaped curve.

b The line of symmetry is approximately at 50, so this is a good estimate of the mean.

49 $P(11 < X < 15)$ can be found on the calculator – either directly or as $\mathrm{P}(X < 15) - \mathrm{P}(X < 11)$. It equals 0.625

50 Some calculators can deal with the given information directly, but some require you to first convert the information into a cumulative probability: $\mathrm{P}(X \leq k) = 0.3$. Using the inverse normal function on the calculator gives $k = 92.1$

51 We first need to convert the original data into ranks. It does not matter whether the ranks are high to low or low to high. We will rank the lowest value as 1:

r_x	1	3	2	4
r_y	3	1	2	4

Then use the calculator to find the Pearson's product moment correlation coefficient of the ranks. The result is 0.2.

52 With tied ranks averaged, the ranks are:

r_x	1	2.5	2.5	4
r_y	2	2	4	2

Using the GDC to find the correlation coefficient of the ranks, the result is 0

53 r_s would be more appropriate as it measures any tendency to increase while r is only looking for linear relationships.

54 r_S, because it is less sensitive to outliers.

55 a $H_0: \mu = 0.4$; $H_1: \mu \neq 0.4$

b $H_0: \mu = 0.4$; $H_1: \mu > 0.4$

Tip: You could write these in words, but it is usually easier to use equations and inequalities where possible. Make sure that the null and alternative hypotheses are in terms of population parameters – in this case, the population mean (μ) rather than the sample mean ($\bar{x}$).

56

blond	brown	black
$100 \times \frac{1}{8} = 12.5$	$100 \times \frac{4}{8} = 50$	$100 \times \frac{3}{8} = 37.5$

Tip: Expected values should not be rounded to the nearest whole number.

57 First work out the probabilities of the binomial distribution, using your GDC:

Outcome	**0**	**1**	**2**
Probability	**0.36**	**0.48**	**0.16**

To find the expected frequencies, multiply each probability by 35:

Outcome	0	1	2
Expected frequency	12.6	16.8	5.6

58 The sample size is 90. The GDC can be used to find the probabilities; however, the ranges must be extended to all possible values otherwise the probabilities would not add up to 1:

x	$x < 120$	$120 \le x < 130$	$130 \le x < 140$	$140 \le x$
Probability	0.309	0.383	0.242	0.0668
Expected frequency	27.8	34.5	21.8	6.01

59 a If you use the calculator to do a chi-squared test, the calculator works out the expected frequencies:

		Variable X		
		A	B	C
Variable Y	D	$\frac{176}{9}$	$\frac{156}{9}$	$\frac{136}{9}$
	E	$\frac{220}{9}$	$\frac{195}{9}$	$\frac{170}{9}$

b The appropriate test is a chi-squared test for independence. This can be done on your GDC. The output is:

$\chi^2 = 9.96$, two degrees of freedom. p-value is 6.86×10^{-3}

Since the p-value is less than 5%, there is significant evidence that the variables are dependent.

60 There are $10 + 15 + 12 + 13 = 50$ observations.

If all the outcomes are equally likely, the expected frequency of each one is $\frac{50}{4} = 12.5$

There are 4 groups of outcomes, so there are $4 - 1 = 3$ degrees of freedom.

All this can be put into the chi-squared goodness of fit test on the GDC to get:

$\chi^2 = 1.04$,

p-value $= 0.792$

Since $0.792 > 0.05$ there is not significance evidence that the outcomes are not equally likely.

Tip: Just because there is not significant evidence that the outcomes are not equally likely, does not mean that there is significant evidence that they are the same.

61 Since χ^2 is a measure of the distance between the observed and expected, a large value represents a big difference. The calculated χ^2 is larger than the critical value so there is significant evidence that the observed frequencies differ from the expected frequencies.

62 Using a one sample, one-tailed t-test from the GDC:

$\bar{x} = 12.2, s_{n-1} = 2.39, t = 2.06, p = 0.0542$

Since $p > 0.05$ there is no significant evidence that the mean is bigger than 10.

63 Using a one sample, two-tailed t-test from the calculator, $t = -3.46$ and p-value $= 5.29 \times 10^{-3}$.

The p-value is less than 5% therefore there is significant evidence that the mean length of newts is not 13 cm.

64 Using a two sample, two-tailed t-test with pooled variance (in this course you always used the pooled variance option):

$t = 0.248, p = 0.814$

Since the p-value is greater than 0.10, there is no significant evidence that the two groups have a different population mean.

65 Using a two sample, one-tailed test with pooled variance:

$t = 2.27, p = 0.0428$

Since the p-value is less than 5%, there is significant evidence that group A is drawn from a population with a larger mean than group B.

66 It assumes that the data is drawn from a normal distribution.

(There are lots of other assumptions, including that the sampling is random and representative and that in the two-sample case the two groups have equal population variance; however, it is the normal distribution one which is focused on in this course.)

5 Calculus

1

x	$\frac{\sin(3x)}{0.2x}$
10	0.25
5	0.2588
1	0.2617
0.1	0.2618

The limit is 0.26

2 Look at the values on the graph close to $x = 2$

The limit is 0.5

3 The derivative is $\frac{dy}{dx} = 12 - 5 = 7$

4 'Rate' means $\frac{dA}{dt}$; 'decreases' means that the rate of change is negative.

$$\frac{dA}{dt} = -kA$$

5 The y value is $f(x)$ and the gradient is $f'(x)$. So when $y = 4$, $f'(x) = -1$.

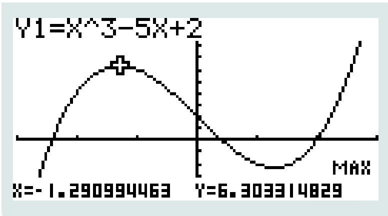
6 Use $\Delta x = x_Q - 4, \Delta y = y_Q - 2$ and gradient $= \frac{\Delta y}{\Delta x}$

x_Q	y_Q	Δx	Δy	Gradient of PQ
5	2.236	1	0.236	0.236
4.1	2.025	0.1	0.025	0.248
4.01	2.002	0.01	0.002	0.250
4.001	2.000	0.001	0.000	0.250

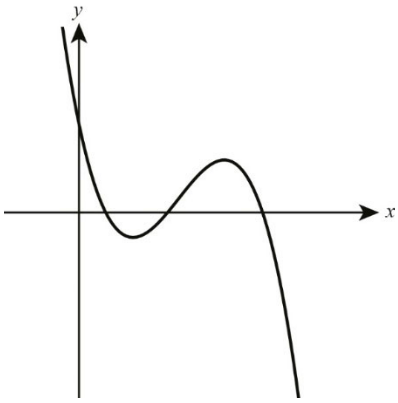
The gradient is ≈ 0.25

7 $f'(x)$ is where the graph is decreasing, which is between the two turning points.

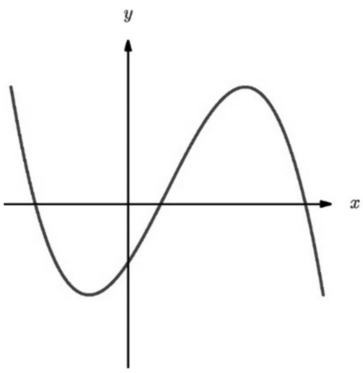
$-1.29 < x < 1.29$



8 The gradient starts positive but decreasing, then changes to negative, then back to positive and then to negative again.



9 The gradient starts off negative, so $f(x)$ is decreasing. It then increases, and then decreases again.



10 $\frac{dy}{dx} = 8x + \frac{1}{2}x^{-6} - 3$

11a $\mathrm{f}(x) = 12x^2 - 3x^6$ so $\mathrm{f}'(x) = 24x - 18x^5$

b $\mathrm{f}(x) = 1 - \frac{3}{2}x^{-4}$, so $\mathrm{f}'(x) = \frac{6}{x^5}$ [or $6x^{-5}$]

c $\mathrm{f}(x) = \frac{4}{5}x - \frac{3}{5} + \frac{1}{5}x^{-1}$, so $\mathrm{f}'(x) = \frac{4}{5} - \frac{1}{5x^2}$ [or $\frac{4}{5} - \frac{1}{5}x^{-2}$]

12 $\mathrm{f}'(x) = 8x + 2x^{-2}$

$$\mathrm{f}'(2) = 16 + \frac{2}{4} = 16.5$$

13 $\frac{\mathrm{d}y}{\mathrm{d}x} = 12 - 5x^{-2}$

$12 - 5x^{-2} = 2$

$$\frac{5}{x^2} = 10, x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

14 $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x = 8, y = 16 - 3 = 13$

Tangent: $y - 13 = 8(x - 4)$

$y = 8x - 19$

15 $\frac{\mathrm{d}y}{\mathrm{d}x} = 3 + 2x^{-2} = 3 + \frac{2}{4} = \frac{7}{2}$

$y = 6 - \frac{2}{2} = 5$

Normal: $y - 5 = -\frac{2}{7}(x - 2)$

$$y = -\frac{2}{7}x + \frac{39}{7}$$

16 $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, so the tangent at $(a, a^2 - 3)$ is:

$y - (a^2 - 3) = 2a(x - a)$

When $x = 0, y = -12$:

$-9 - a^2 = -2a^2$

$a = \pm 3$

17 Use GDC to find the gradient and to draw the tangent.

a gradient = -0.021

b $y = -0.021x + 0.13$

18 Use GDC to sketch the graph of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and intersect it with $y = 2$. The coordinates are $(0.5, -0.098)$.

19 $3x^3 - 3x^{-2} + c$

20 $\int \frac{1}{2}x^3 - \frac{3}{2}x^{-2}\mathrm{d}x = \frac{1}{8}x^4 + \frac{3}{2}x^{-1} + c$

21 Use GDC: $\int_2^3 2x^3 - 1\,\mathrm{d}x = 31.5$

22 Integrate: $y = \int 4x + 2\,\mathrm{d}x = 2x^2 + 2x + c$

Use $y = 3, x = 2$: $3 = 2(2^2) + 2(2) + c \Rightarrow c = -9$

So $y = 2x^2 + 2x - 9$

23 $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 2ax = 0$

$2x(3x - a) = 0$

$x = 0$ or $\frac{a}{3}$

24 Sketch $y = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{x} + \sqrt{x}\right)$ and find the intersection with the x-axis.

$x = 2.52$

25 From the graph: $(-0.281, 2.12)$.

26 Use the graph, checking the local minimum $(3.46, -33.3)$ and the end point $(-5, -62.5)$. The smallest value is -62.5

27 $S = x^2 + 4 \times \left(x \times \frac{32}{x^2}\right) = x^2 + \frac{128}{x}$

Draw the graph for $x > 0$ to find that the minimum value is 48 cm^2.

28 Using the trapezoidal rule with $h = 0.5$:

$$\int_2^5 \mathrm{f}(x)\mathrm{d}x \approx 0.25[1.6 + 0.8 + 2(2.1 + \cdots + 1.5)] = 5.65$$

29 Use GDC to create a table, with x-values spaced by $\frac{2}{5} = 0.4$.

x	0	0.4	0.8	1.2	1.6	2
y	0	0.3031	0.4595	0.5223	0.5278	0.5

Trapezoidal rule with $h = 0.4$÷

$$\text{Area} \approx 0.2\{0 + 0.5 + 2(0.3031 + 0.4595 + 0.5223 + 0.5278)\} = 0.825$$